Exercise sheet 7

This exercise sheet should be completed individually or in pairs. The solutions will be discussed on Thursday, 25.06.2026, where guidance on the tasks will be provided.

Students encountering difficulties are welcome to arrange an individual online appointment by contacting: ianto.cannon@zarm.uni-bremen.de

Turbulent flow data

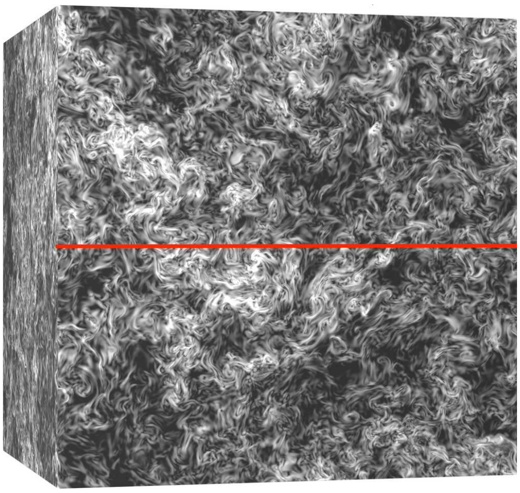

Figure 1: Schematic of the turbulent dissipation in a cubic domain containing homogeneous isotropic turbulence. The red line indicates the sampled line through the flow field. It is in the x direction.

We will analyse publicly available simulation data downloaded from the Johns Hopkins Turbulence Database: https://turbulence.idies.jhu.edu/home. You can find the dataset `iso1024.zip` on the e-learning platform. It contains eight files of the form `velocity_T.txt`, where T denotes the time index. Each file contains three columns corresponding to the velocity components (u, v, w) sampled along a line through the computational domain (see Fig. **??**).

- The domain is periodic in all three directions x, y, z, and has side length 2π m.
- Resolution: $N^3 = 1024^3$ grid points.
- Kinematic viscosity: $\nu = 1.85 \times 10^{-4}\ \mathrm{m^2\,s^{-1}}$.
- Time step: $\Delta t = 2.0 \times 10^{-3}$ s.
- Forcing: Large-scale forcing in Fourier space for modes $0.5\ \mathrm{m^{-1}} \leq k \leq 2.5\ \mathrm{m^{-1}}$.

A: Velocity

1. Unzip `iso1024.zip` and load the velocity files into Python using:

```
import numpy as np
import glob
files = glob.glob("velocity_*.txt")
u_all, v_all, w_all = [], [], []
for f in files:
    data = np.loadtxt(f, comments="#")
    u_all.append(data[:, 0])
    v_all.append(data[:, 1])
    w_all.append(data[:, 2])
u_all = np.array(u_all)
v_all = np.array(v_all)
w_all = np.array(w_all)
```

2. Plot the velocity component u along the sampling line for all time instances.

3. Plot v and w and compare their magnitudes qualitatively with u.

B: Dissipation

1. The energy dissipation rate in an incompressible fluid is

$$\epsilon = 2\nu S_{ij}S_{ij}, \qquad S_{ij} = \frac{1}{2}\left(\partial_i u_j + \partial_j u_i\right), \tag{1}$$

where ν is the kinematic viscosity. Under the assumption of isotropy, this can be written as

$$\epsilon(t) = 15\nu\left\langle\left(\frac{\partial u}{\partial x}\right)^2\right\rangle_x. \tag{2}$$

Compute $\epsilon(t)$, averaged over space. You should obtain a value of around $0.1\ \mathrm{m^2s^{-3}}$.

2. Using dimensional analysis, derive an expression for the Kolmogorov length scale η in terms of ϵ and ν.

3. Compute $\eta(t)$.

4. Is the simulation sufficiently resolved at the smallest scales?

C: Kinetic energy

1. Compute the mean kinetic energy,

$$E(t) = \frac{1}{2}\left\langle u_i u_i\right\rangle_x. \tag{3}$$

2. Compute the Taylor-scale Reynolds number as a function of time.

3. Compute the Fourier transform of the velocity field along x using numpy;

```
N = u_all.shape[1]
u_hat = np.fft.rfft(u_all, axis=1) / N
v_hat = np.fft.rfft(v_all, axis=1) / N
w_hat = np.fft.rfft(w_all, axis=1) / N
```

4. For each of the 8 snapshots in time, plot the energy spectrum $E(k,t)$ on a log-log scale and verify that its integral matches the mean kinetic energy.

5. Compute the time-averaged spectrum $E(k)$.

6. The energy spectrum decays to machine precision (approximately 10^{-17}) for wavenumbers $k \gtrsim N/3$. Replot the spectrum excluding these modes[1].

7. Add the Kolmogorov scaling law $k^{-5/3}$ to the plot for comparison.

8. Identify the forcing, inertial, and dissipation ranges in spectral space. Mark them on your spectrum.

D: The energy cascade

1. Describe the Richardson energy cascade in homogeneous turbulence.

2. Explain why the dissipation rate remains finite as viscosity tends to zero.

3. If the largest eddies in the flow have size L and velocity scale U, what is their characteristic turnover time scale?

4. Explain why the velocity samples used to compute the energy spectrum were taken 500 time steps apart, and relate this to statistical independence.

[1] Modes with $k \gtrsim N/3$ are affected by numerical aliasing, whereby nonlinear interactions generate unresolved high-wavenumber modes that are folded back into the resolved range by the discrete Fourier transform. See also wikipedia.org/wiki/Aliasing or Pope (2000), *Turbulent Flows*, Appendix F.

Exercise sheet 8

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Channel flow data

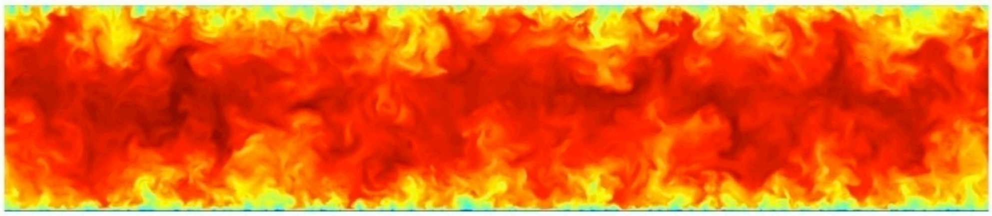

Figure 1: Streamwise velocity in a slice through the channel. Credit to the John Hopkins Turbulence Database.

This dataset originates from the Johns Hopkins Turbulence Database. The file `channel.zip` contains velocity fields $u(x,y)$, $v(x,y)$, and $w(x,y)$ sampled on a two-dimensional slice of a turbulent channel flow.

Data Structure

- Streamwise direction (x): Mapped along the rows of `u.txt`, `v.txt`, and `w.txt`.
- Wall-normal direction (y): Mapped down the columns of `u.txt`, `v.txt`, and `w.txt`.
- Grid spacing: Non-uniform in the y-direction, with finer resolution near the walls.
- Coordinates: The file `y.txt` contains the specific wall-normal (y) coordinates.

Simulation details

- Channel half-height: $h = 1\text{m}$
- Streamwise spacing: $\Delta x = 0.196\text{m}$
- Kinematic viscosity: $\nu = 5 \times 10^{-5}\text{m}^2\text{s}^{-1}$
- Mean pressure gradient: $\tau_w = \frac{\mathrm{d}P}{\mathrm{d}x} = 0.0025\text{kg}\,\text{m}^{-2}\,\text{s}^{-2}$
- Fluid density: $\rho = 1\text{kg}\,\text{m}^{-3}$

A: Bulk flow rate

1. Unzip `channel.zip` and load the data using:

```
y = np.loadtxt("y.txt").squeeze()
u = np.loadtxt("u.txt")
v = np.loadtxt("v.txt")
w = np.loadtxt("w.txt")
```

2. Plot a colour map of $u(x, y)$ and verify that the no-slip boundary condition is satisfied at the walls, i.e. $u \approx 0$ at $y = \pm h$.

3. Compute the mean velocity profile $U(y)$ by averaging in the streamwise direction:
$$U(y) = \langle u(x, y)\rangle_x, \tag{1}$$
and plot $U(y)$ across the channel height.

4. Using the trapezoidal rule, compute the bulk flow rate (per unit spanwise length)
$$Q = \int_{-h}^{h} U(y)\,\mathrm{d}y, \tag{2}$$
e.g. using `numpy.trapz`. Hence compute the bulk velocity $U_b = Q/(2h)$ and the bulk Reynolds number
$$Re_b = \frac{2hU_b}{\nu}. \tag{3}$$

B: Forces in the channel

1. Using the wall shear stress τ_w given above, compute the friction velocity and the friction Reynolds number
$$u_\tau = \sqrt{\tau_w/\rho}, \qquad Re_\tau = \frac{u_\tau h}{\nu}. \tag{4}$$

2. Compute and plot the viscous shear stress
$$\tau_\nu(y) = \rho\nu\frac{dU}{dy} \tag{5}$$
as a function of the distance from the wall. Comment on its behaviour near the wall and in the channel core.

3. Compute the Reynolds shear stress
$$\tau_{\mathrm{Re}}(y) = -\rho\langle u'v'\rangle_x, \tag{6}$$
where fluctuations are defined as $u' = u - U(y)$ and $v' = v - V(y)$, with $V(y) \approx 0$. Plot viscous and Reynolds shear stresses together.

4. Starting from the Reynolds-averaged Navier–Stokes (RANS) equations for statistically stationary, fully-developed channel flow, derive the total shear stress relation
$$\tau_{\mathrm{tot}}(y) = \tau_\nu(y) + \tau_{\mathrm{Re}}(y) = \tau_w\frac{y}{h}. \tag{7}$$

5. Compute and plot $\tau_{\mathrm{tot}}(y)$ and compare it with the theoretical linear profile derived above. Discuss possible sources of deviation between theory and numerical data.

Optional extra: Law of the wall

1. Using the friction velocity u_τ computed above, express the data in wall units:

$$y^+ = \frac{y u_\tau}{\nu}, \qquad U^+ = \frac{U}{u_\tau}. \tag{8}$$

   Plot U^+ versus y^+ on a semi-log scale.

2. Plot also the logarithmic law of the wall:

$$U^+ = \frac{1}{\kappa}\ln(y^+) + B, \tag{9}$$

   with $\kappa = 0.41$ and $B = 5.2$. Identify the range of y^+ where this relation holds.